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Nonlinear excitations in classical ferromagnetic chains

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Abstract. General single solitary-wave excitations are determined for the classical continuum Heisenberg chain in the presence of external magnetic and anisotropy fields. These include both domain walls and pure solitons as examples. Conditions for propagation are carefully analysed. The complete integrability of the zero-anisotropy limit is suggested as a basis for (i) controlled singular perturbation theory and (ii) formulation of classical statistical mechanics in a natural configurational (nonlinear normal mode) representation.

1. Introduction

There is a developing interest (Nakamura and Sadada 1974, Lakshmanan *et al* 1976, Lakshmanan 1977, Tjon and Wright 1977, Lamb 1976, 1977, Takhtajan 1977) in the one-dimensional classical¶ Heisenberg ferromagnet and particularly in fully nonlinear solutions to the spin equations of motion in the continuum limit. Most interestingly, it has been possible (in the continuum limit) to solve the arbitrary initial value problem exactly since the equations of motion for the spin variables themselves (Takhtajan 1977) (or for the energy and momentum densities (Lakshmanan 1977, Lamb 1976)) are soluble by the inverse spectral transform method: the continuous Heisenberg chain is completely integrable. The energy and momentum density excitations take the form of strict envelope soliton solutions (Scott *et al* 1973, Bullough *et al* 1978), which are simply related to those of the nonlinear Schrödinger equation, and the general solution follows from the general N-soliton solution.

The complete solutions indicated above are so far limited to the isotropic Heisenberg chain. However, it is well known (Bishop *et al* 1977) that the presence of local or exchange anisotropy will admit a rather different type of excitation, namely a *domain wall.* Experimentally and theoretically relevant questions arise concerning the possible co-existence of both solitons and walls and of more general composite excitations, particularly in view of accessible quasi-one-dimensional classical spin systems (Steiner *et al* 1976) (although the discussion applies formally to planar excitations in three dimensions).

In this paper we begin a study of these questions by considering the possible single solitary-wave solutions to the general continuum Heisenberg ferromagnetic chain, including uniaxial anisotropy and an external magnetic field. These solutions certainly

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¶ We will not consider solutions of the *quantum* (spin-$\frac{1}{2}$) X–Y–Z model which is also of current interest as a technical device for solving equivalent models in statistical mechanics and field theory (e.g. Luther 1976).

do not exhaust all excitation types, and it is intended to report on others in future publications. Nevertheless, the present analysis serves to clarify current literature and provides some suggestive results concerning the compatibility of soliton and wall excitations.

The equation of motion we study is (Landau 1965)

$$\partial \boldsymbol{S}/\partial t = J\boldsymbol{S}\times(\partial^2\boldsymbol{S}/\partial x^2)+\gamma\boldsymbol{S}\times\boldsymbol{\Omega}_0+(K/S^2)\boldsymbol{S}\times\boldsymbol{n}S_z, \qquad (1.1)$$

where $\boldsymbol{S}(x, t)$ is the magnetisation, $\boldsymbol{\Omega}_0$ is the applied magnetic field, $\boldsymbol{n}$ is a unit vector in the z-direction, J is the exchange constant, K/S^2 is the (local uniaxial) anisotropy constant, and γ is the gyromagnetic ratio.

In § 2 we show that equation (1.1) has single solitary-wave solutions and find their analytic form in various cases, including the strict solitons when $K = 0$. In § 3 we show domain walls propagating with a constant velocity are allowed only under special circumstances and only when a (phenomenological) damping is included in equation (1.1). The work of Walker (1963) and Enz (1964) on propagating walls is critically re-examined and formulated in the context of nonlinear physics emphasised here. Section 4 contains our conclusions and further discussion, particularly of possible incorporation of these nonlinear modes in classical statistical mechanics.

2. Single solitary waves: domain walls and solitons

Following the analysis of Tjon and Wright (1977) we write the Hamiltonian density from which equation (1.1) derives in spherical coordinates as

$$\mathcal{H}(x)=\tfrac{1}{2}JS^2[(\partial U/\partial x)^2(1-U^2)^{-1}+(\partial\phi/\partial x)^2(1-U^2)+S\Omega_0(1-U)]. \qquad (2.1)$$

Here $S_1 = S\sin\theta\cos\phi$, $S_2 = S\sin\theta\sin\phi$, $S_3 = S\cos\theta = SU(x)$, and Ω_0 is the applied magnetic field, where γ has been absorbed in Ω_0. The linear and angular momentum operators are constants of the motion, and we are free to impose the constraints

$$P = P_0, \qquad M = M_0, \qquad (2.2)$$

where

$$P = S\int_{-\infty}^{+\infty} \mathrm{d}x\,(1-U)\frac{\partial\phi}{\partial x}, \qquad (2.3)$$

$$M = S\int_{-\infty}^{+\infty}\mathrm{d}x\,(U-1). \qquad (2.4)$$

Introducing Lagrangian multipliers Ω_1 and ν, the governing equations of motion can now be obtained by minimising

$$I(\Omega_1, \nu; U, \phi) = \int \mathcal{H}(x)\,\mathrm{d}x - \Omega_1(M - M_0) - \nu(P - P_0) \qquad (2.5)$$

with respect to (U, ϕ), subject to the constraints (2.3) and (2.4). Ω_1 and ν will play the roles of angular and linear velocity in §2.3 below, where we exhibit solutions in the essentially solitary-wave form

$$U(x, t) = U(x-\nu t),$$
$$\phi(x, t) = \Omega_1 t + \bar{\phi}(x - \nu t). \qquad (2.6)$$

For the moment consider the equations of motion following from the minimisation of (2.5):

$$0 = JS^2 \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}(1-U^2)\right) + S\nu \frac{\partial U}{\partial x}, \tag{2.7}$$

$$0 = JS^2\left[\frac{\partial^2 \theta}{\partial x^2}(\sin\theta)^{-1} - U\left(\frac{\partial \phi}{\partial x}\right)^2\right] - S(\Omega_0+\Omega_1) + S\nu\frac{\partial \phi}{\partial x}. \tag{2.8}$$

Including a local anisotropy term (cf (1.1)) changes (2.8) to

$$0 = JS^2\left[\frac{\partial^2 \theta}{\partial x^2}(\sin\theta)^{-1} - \cos\theta\left(\frac{\partial \phi}{\partial x}\right)^2\right] - S(\Omega_0+\Omega_1) + S\nu\frac{\partial \phi}{\partial x} - 2K\cos\theta, \tag{2.9}$$

where $K = \frac{1}{2}JS^2\tau$. We have yet to specify boundary conditions which will allow us to determine unique solutions to (2.7) and (2.8) or (2.9). Since we are interested in studying both domain wall and pulse solitary-wave solutions, we integrate (2.7) and (2.9) leaving the integration constants as arbitrary.

Integrating (2.7) once gives

$$0 = -JS^2(\partial\phi/\partial x)(1-U^2) - S\nu U + a_1, \tag{2.10}$$

with the integration constant a_1 given by

$$\partial\phi/\partial x = (a_1 - S\nu U)/JS^2(1-U^2). \tag{2.11}$$

If $U = \pm 1$ and $x \to -\infty$, then $a_1 = \pm S\nu$. If $\phi_x = 0$ and $\nu = 0$, then $a_1 = 0$. If $\nu \neq 0$, then $\phi_x \neq 0$, since $a_1 \neq S\nu(x)$ because a_1 is constant. (We omit the trivial case U = constant.) Substituting (2.11) into (2.9) and multiplying by $\sin\theta\theta_x$, we integrate again to obtain

$$\left(\frac{\partial \theta}{\partial x}\right)^2 = -\frac{a_1 S\nu}{J^2S^4}\left(\frac{1-U}{1+U}\right) + \frac{1}{2J^2S^4}\frac{(a_1-S\nu)^2}{1-U^2} - \Omega U - \frac{1}{2}\tau\cos 2\theta + a_2, \tag{2.12}$$

where a_2 is a second integration constant and we have used the notation $\Omega \equiv \Omega_0 + \Omega_1$.

If $a_1 = S\nu$ and $\theta \to 0$ as $x \to \infty$, then (2.12) gives ($\theta_x \equiv \partial\theta/\partial x$)

$$\theta_x^2 = -V^2\left(\frac{1-U}{1+U}\right) - \Omega U - \frac{1}{2}\tau(1-2\sin^2\theta) + a_2, \tag{2.13}$$

where $V \equiv \nu/JS$. With the further boundary condition $\theta_x = 0$ and $\theta = 0$, as $x \to -\infty$, we have $a_2 = \Omega + \frac{1}{2}\tau$, so that (cf Tjon and Wright 1977)

$$\theta_x^2 = \frac{4\Omega(1-U)}{1+U}\left(\frac{1}{2}(1+U) - \frac{V^2}{4\Omega} + \frac{(1+U^2)\tau}{4\Omega}\right). \tag{2.14}$$

If we seek to impose 180° domain wall boundary conditions, however, we see that $\theta = \pi(U=-1)$, for $x \to +\infty$ implies $|\theta_x| \to \infty$. We thus conclude immediately that there are no single 180° domain wall solutions in general. Such excitations are in fact possible only in one special limit (see § 2.1 below). The point we wish to emphasise here is that the factor $(1-U^2)^{-1}$ in (2.11) excludes domain solutions unless $V = 0$ and $\phi_x = 0$.

2.1. Static domain wall: anisotropy field

One limit in which domain solutions are very familiar (Bishop *et al* 1977) is for $V = 0$ and $\Omega = 0$. Then (2.14) becomes

$$\theta_x^2 = (1-U)(1+U)\tau \tag{2.15}$$

and the troublesome denominator is removed. Thus θ_x can be zero as $x \to -\infty$ ($U = +1$) and also as $x \to +\infty$ ($U = -1$). Integration follows simply upon the substitution $\beta = \frac{1}{2}\theta$ and $y = \tan\beta$, giving

$$y = \exp(x\tau^{1/2}) = \tan(\tfrac{1}{2}\theta), \tag{2.16}$$

i.e.

$$\theta = \cos^{-1}\tanh(\pm x\tau^{1/2}). \tag{2.17}$$

This is a static 180° domain wall solution (see figure 1).

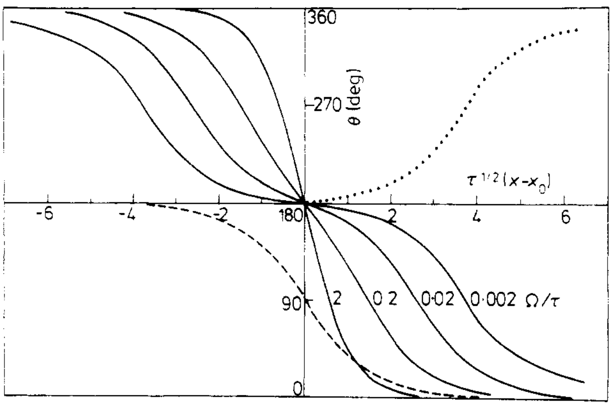

Figure 1. Zero-velocity 2π-walls with finite magnetic (Ω) and anisotropy (τ) fields (§ 2.2). Also included for comparison are the reflected solution for $\Omega = 0{\cdot}002\tau$ (cf π-pulse, § 2.3) and the π-wall with no magnetic field (§ 2.1). The latter should be compared with a displaced form of the finite-Ω curves. Full curve: static π- and 2π-walls (symmetric about x_0). Dotted curve: reflected solution for $\Omega/\tau = 0{\cdot}002$ (§ 2.4). Broken curve: π-wall ($\Omega = 0$) (§ 2.1).

2.2. Magnetic and anisotropy fields: static solution

If $V = 0$ but $\Omega, \tau \neq 0$, we can use the same substitutions as in §2.1 to express (2.14) as

$$4\beta_x^2 = \theta_x^2 = 4\Omega\sin^2\beta + 4\tau\sin^2\beta\cos^2\beta. \tag{2.18}$$

With $y = \tan\beta$ we find easily

$$y = a\ \mathrm{cosech}(\pm a\Omega^{1/2}x),$$

i.e.

$$\theta = 2\tan^{-1}[a\ \mathrm{cosech}(\pm a\Omega^{1/2}x)], \tag{2.19}$$

with

$$a = (1+\tau/\Omega)^{1/2}.$$

Note how this solution is apparently in a quite different sector of solution space than (2.17). It is a symmetric 2π-wall with an interesting internal structure (see figure 1). One way of viewing the solution is as a state of two (contracted) single π-walls (§2.1) locked together by the field Ω. For as $\Omega/\tau \to 0$, the shelf length $w_1 \to \infty$ and the single wall width w_2 tends to the wall width in §2.1: we thus have two infinitely separated conventional π-domain walls. In general w_1 and w_2 are both functions of τ and Ω. However, in the opposite limit of weak anisotropy, $\tau/\Omega \to 0$, $w_1 \to 0$ and w_2 tends to a characteristic width for the stationary limit of a strict soliton pulse which we derive below (§ 2.3). This composite wall structure is restricted to $V = 0$ (see § 2.4) (at least as a pure solitary wave). On symmetry grounds we might expect an alternative structure of a *pulse* of amplitude π (see broken curve in figure 1). This solution does indeed exist and is able to move (thereby allowing a continuous deformation), as we will show in § 2.4). Such a structure will be viewed (at $V = 0$) as a bound state π-wall and π-anti-wall.

The ability of the field to stabilise a two-π-wall state is interesting. In the absence of any lattice discreteness two walls of type (2.17) (i.e. zero field) are well known (Rubinstein 1970) to suffer a mutual exponentially short-ranged repulsion, so that they can be static only at infinite separation. The finite field overcomes this repulsion and stabilises the wall at increasingly short separation as Ω/τ increases. Further, the field's main effect is to distort the two-wall solution near $\theta = \pi$ (so as to satisfy boundary conditions). Indeed the gradient at $\theta = \pi(\mathrm{d}\theta/\mathrm{d}x = -2\Omega^{1/2})$ is totally independent of τ. The distorted region increases with Ω/τ, but for very small Ω/τ much of the solution is quantitatively identifiable as displaced π-walls (see figure 1).

2.3. *Zero anisotropy*: *strict solitons*

The zero-anisotropy field limit $\tau = 0$, $V \neq 0$, $\Omega \neq 0$, was actually first solved by Hasimoto (1972, Lamb 1976, 1977), because the problem of vortex filament motion treated by him is formally identical with the spin problem here, where the torsion vector is identified with the spin vector. The *single*-soliton solution, however, follows quite directly from (2.14).

With $\tau = 0$ the substitution $y = \sin\beta$ in (2.14) gives (cf Tjon and Wright 1977)

$$y = \sin\beta = b\ \mathrm{sech}[b\Omega^{1/2}(x - x_0 - \nu t)], \tag{2.20}$$

with

$$b^2 = 1 - V^2/4\Omega = 1 - \cos^2\beta_0, \tag{2.21}$$

where $2\beta_0$ is the maximum θ-variation at the centre of the translating pulse (2.20). Thus

$$\cos\theta(x, t) = 1 - 2b^2\,\mathrm{sech}^2[b\Omega^{1/2}(x - x_0 - \nu t)]. \tag{2.22}$$

The θ-variation is *necessarily coupled* to a ϕ-variation. For, using (2.11) with $a_1 = S\nu$, we find

$$\phi(x, t) = \phi_0 + \Omega_1 t + \tfrac{1}{2}V(x - x_0 - \nu t) + \tan^{-1}\{b(1 - b^2)^{-1/2}\tanh[b\Omega^{1/2}(x - x_0 - \nu t)]\}. \tag{2.23}$$

Tjon and Wright (1977) have shown that the constants of motion for this coupled nonlinear solution are

$$M_0 = 4bS\Omega^{1/2}, \qquad P_0 = 4S\beta_0.$$

Also the energy

$$E \equiv \int dx\, \mathscr{H}(x) = 4JS^2 b\Omega^{1/2} + 4JSb\Omega_0\Omega^{-1/2}.$$

We note that the θ-pulse (2.22) has increasing width, decreasing amplitude and decreasing energy as $b \to 0$ (cf figure 2(*a*)). In this it is similar to the so-called 'breather' solutions to another fashionable nonlinear equation, the sine–Gordon equation (Bullough *et al* 1978). This is not surprising because the high-frequency (small-amplitude) breather solutions can be shown (Kaup and Newell 1978) to be equivalent to the soliton solutions of the nonlinear Schrödinger equation to which the excitation (2.22), (2.23) is directly related (Lakshmanan 1977, Lamb 1977, Takhtajan 1977) (see

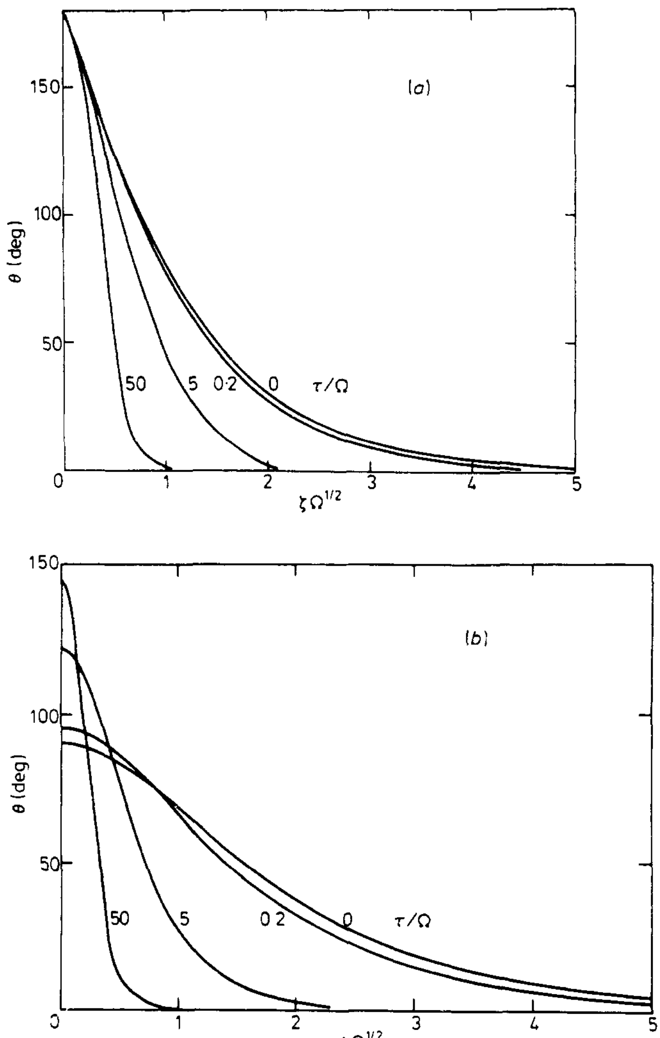

Figure 2. Finite anisotropy (τ) π-pulse solutions (§ 2.4): zero velocity, $V = 0$; (*b*) finite velocity, $V^2 = 2\Omega$. The solutions are symmetric about the origin ($\zeta = 0$). Note that the singularity at $\zeta = 0$ is removed in case (*b*).

below). Furthermore the breather solution can also be viewed as a bound state of soliton and anti-soliton (cf the remarks above). The ϕ-variation (additional to the rotation Ω_1) also tends to zero as $b \to 0$, and the translation velocity ν maximises ($\nu \to 4JS^{1/2}$). The analogue to the latter in the sine–Gordon breather case is rather that an internal oscillation frequency maximises—translation velocity and amplitude are independent, and the present soliton differs in this sense. It is interesting to note the nonlinear dispersion exhibited by the present pulse soliton (Lakshmanan *et al* 1976, Tjon and Wright 1977):

$$E = 8JS^3M_0^{-1}(1-\cos(P_0/2S)] + M_0\Omega_0. \tag{2.24}$$

As in the sine–Gordon case this sector of solution space is distinct from the continuum mode ('linear magnon') sector, but also merges continuously with it in the limit $b \to 0$.

The large-amplitude ($\beta_0 \to \pi/2$, $b \to 1$) limit of (2.22) (cf figure 2(b)) corresponds to a narrow pulse (width $\sim\Omega^{-1/2}$) with $V \to 0$ and maximum energy and magnetisation M. Similarly the corresponding ϕ-variation (2.23) is greatest. Although singular for $V = 0$ ($\mathrm{d}\theta/\mathrm{d}x$ is discontinuous), this is an especially interesting regime to study analytically and numerically when $\tau \neq 0$. We have already seen one possible continuation to $\tau \neq 0$ in § 2.1. (Note that $\mathrm{d}\theta/\mathrm{d}x$ at $x = 0$ agrees with (2.19).) A second possibility will be described in § 2.4 which is also valid for $V \neq 0$.

The single-soliton solution is readily shown to agree with the result obtained by Hasimoto with suitable re-identifications of parameters (see equation (3.19b) of Hasimoto (1972)). More importantly the solution is now known to be a *strict soliton* in a precise mathematical sense (Bullough *et al* 1978), since the spin equation of motion in the isotropic limit has been shown (Lakshmanan 1977, Lamb 1977, Takhtajan 1977) to be exactly soluble by the canonical 'inverse spectral transform' or related techniques (e.g. as a Riemann problem (Zhakarov and Manakov 1979)) (see also § 4). The most direct demonstration has been given by Takhtajan (1977): the parameters (x_0, ν, Ω_1, ϕ_0) required for complete specifications of solution (2.22), (2.23) then arise as the asymptotic scattering data of an associated linear eigenvalue problem, providing a *natural* new set of dynamical variables or generalised action–angle variables (cf Bullough *et al* 1978). (The external magnetic field can be scaled out, as is implicit in the form $\Omega = \Omega_0 + \Omega_1$.) Further, this result means that an *arbitrary* solution (subject to weak boundary conditions) can always be decomposed in to a finite number of identifiable nonlinear normal modes (solitons and a continuum sector), depending only upon the initial conditions. We are thus no longer limited to the single-soliton solution above, but interactions between component nonlinear modes are of a purely asymptotic phase-shifting form and therefore surprisingly tractable. This remarkable feature has been used elsewhere (see Bishop 1978b) to develop the statistical mechanics of other totally integrable nonlinear Hamiltonian systems. Similarly here, it provides the means of formulating the statistical mechanics of the classical Heisenberg chain, as will be described elsewhere.

2.4. *The general solitary-wave solution*

If we retain finite anisotropy as well as the conditions for § 2.3, we can find an *analytic* solitary-wave solution to (1.1) as follows. From (2.14) with $\beta = \frac{1}{2}\theta$, we have

$$\beta_x^2 = \Omega \tan^2\beta\left(\cos^2\beta - \frac{V^2}{4\Omega} + \frac{\tau}{\Omega}\cos^4\beta\right). \tag{2.25}$$

Putting $z=\sin^2\beta$ (substitution of $z=\tan^2\beta$ is equally convenient), (2.25) reduces to

$$dx = \pm(dz/2z)[\Omega(1-z)-\tfrac{1}{4}V^2+\tau z^2]^{-1/2}, \tag{2.26}$$

where we have introduced boundary conditions $\cos\theta=+1$ as $|x|\to\infty$. Form (2.26) can be integrated (Gradshteyn and Rhyzik 1965) once more, with the result

$$\tanh[\pm 2(x-x_0-\nu t)(\Omega+\tau-\tfrac{1}{4}V^2)^{1/2}] = \frac{2(\Omega+\tau-\frac{1}{4}V^2)^{1/2}[\tau\sin^4\beta-(\Omega+2\tau)\sin^2\beta+(\Omega+\tau-\frac{1}{4}V^2)]^{1/2}}{2(\Omega+\tau-\frac{1}{4}V^2)-(\Omega+2\tau)\sin^2\beta}. \tag{2.27}$$

This form is valid for $(\Omega+\tau-\frac{1}{4}V^2)>0$; otherwise the tanh becomes tan. The propagation velocity is thus limited to $|V|<2(\Omega+\tau)^{1/2}$. Equation (2.27) can be solved explicitly in the surprisingly simple form

$$\sin^2(\tfrac{1}{2}\theta)=2\alpha^2/[\gamma+(\Omega^2+\tau V^2)^{1/2}\cosh(\pm 2\alpha\zeta)], \tag{2.28}$$

where we have used the notation

$$\alpha=(\Omega+\tau-\tfrac{1}{4}V^2)^{1/2}, \qquad \gamma=\Omega+2\tau, \qquad \zeta=x-x_0-\nu t,$$

and omitted unphysical solutions to (2.27).

Both (2.2) and § 2.3 agree with (2.28) in their respective limits $V\to 0$ and $\tau\to 0$. We can now learn much more, however. The solitary-wave solution is evidently symmetric (modulo π) about $\zeta=0$, and we find easily that $\theta(\zeta=0)=\pm\pi$ only if $V=0$. Furthermore it is readily shown that $\zeta=0$ is the only point for which $\theta(\zeta)=\pm\pi$. Thus 2π-walls as in § 2.2 can only be static as pure and constant-velocity solitary waves. The pulse soliton, on the other hand, can move and is to be considered the continuous deformation of the pure soliton pulse ($\tau=0$) of § 2.3, for which numerical evidence has been given by Tjon and Wright (1977). (The bifurcation at $V=0$ might be related to partial signatures of multiple modes which they also observed numerically.) For $V=0$ and $\tau/\Omega\neq 0$ the pulse amplitude is still π but its shape is modified, although the singularity at the pulse centre remains (figure 2(*a*)) with exactly the same gradient $(\pm 2\Omega^{1/2})$ (cf § 2.2). (Note that the 2π-wall solution removes this discontinuity (figure 1).) For $V\neq 0$ the pulse amplitude is increased ($\tau>0$) towards π and we find

$$\sin^2\beta(\zeta=0,\tau\neq 0)=(2\tau)^{-1}[\gamma-(\gamma^2-4\tau\alpha^2)^{1/2}] = \sin^2\beta(\zeta=0,\tau=0)+(V^4/16\Omega^2)\tau/\Omega+O(\tau/\Omega)^2. \tag{2.29}$$

It is again instructive to view this finite-anisotropy solution as a state of two conventional π-walls (§ 2.1) stabilised by the field. In contrast to § 2.2, however, we are here concerned with a composite wall and anti-wall structure. For $\Omega=0$ these suffer an attractive short-range interaction (Rubinstein 1970) and would collapse, but the finite field allows them to stabilise at finite separation with some local distortion (and to translate with a uniform centre-of-mass motion). As τ/Ω increases, the separation now *decreases* (contrast § 2.2), and the region of the structure (around its centre) which deviates from the pure π-wall (anti-wall) also decreases. In fact the structure is essentially that of the π-wall for

$$\cosh(2a\zeta)\gg(1+2\tau/\Omega)/(1+\tau V^2/\Omega^2)^{1/2}. \tag{2.30}$$

For all V, as $\tau/\Omega\to\infty$, $\theta(\zeta)\to 0$ except for $\theta(0)\to\pi$ (cf (2.29). As $\tau/\Omega\to 0$, the pulse widens and its width is controlled by $\Omega^{-1/2}$ as in § 2.3 rather than $\tau^{-1/2}$ (§ 2.1). We note again that the limiting gradient at the pulse centre ($\zeta = 0$) is independent of τ even in the singular limit $V = 0$: the field always controls behaviour sufficiently near $\theta = \pi$. These features are all summarised in figure 2. The ϕ-variation corresponding to (2.23) follows directly from (2.10) and (2.11).

We have not yet examined the stability of these various solutions. The pulse mode is likely to be stable in view of its known stability at $\tau = 0$ ($V \neq 0$). The 2π-wall is less clear. It may be susceptible to break-up into π-walls (and spin waves), but the present solitary-wave analysis does not exclude an accelerating motion (cf §3) or motion with internal dynamics.

3. Dynamic domain walls

We have seen in § 2.1 that constant-velocity π-domain wall solutions to (1.1) exist only if $V = 0$ and $\Omega_0 = 0$. A collection of such walls may simply be unstable towards collapse into pulse solitons and spin waves even with finite anisotropy (τ), as is proven to be the case for $\tau = 0$ and boundary conditions $\theta \to 0$ as $|x| \to \infty$ (Takhtajan 1977). Certainly a continuously deformed pulse solitary wave branch exists (equation (2.27)) for $\tau \neq 0$, but so far we have only considered a single solitary wave, and more interesting general solutions will need to be investigated. The absence of single-wall solutions for $V \neq 0$ or $\Omega_0 \neq 0$ followed directly from (2.14) because we could no longer satisfy the appropriate boundary conditions: $\theta_x(\pm\infty) = 0$, $\theta(+\infty) = \pm\pi$, $\theta(0) = 0$. Physically the point is that these conditions can be satisfied when there are terms with π-degeneracy (i.e. $\tau \cos 2\theta$) in the equation of motion, but not with a 2π-period (e.g. $\Omega_0 \cos\theta$). In the first case θ_x (equation (2.14)) is proportional to $(1-\cos\theta)(1+\cos\theta)$ which can be zero at $\theta = 0$ and π, whereas in the second case θ_x only contains a term proportional to $1-\cos\theta$, and the second boundary condition cannot be satisfied.

The dynamics of π-domain walls are nevertheless of very practical concern in real materials. Therefore in this section we examine the two most popular prototype treatments of wall dynamics due to Walker (1963) and Enz (1964). Our main purpose is to cast these two approaches into the logical framework adopted in § 2, so that we can see clearly the way in which they avoid the problem exposed there, and also to define their approximations and limitations. Again we consider only single-wall solutions, and the general solution remains to be explored.

(*a*) *Walker's solution* (1963)

Here the wall is assumed to lie in the yz plane, θ is the angle $\boldsymbol{S}$ makes with the z axis and ϕ is the angle of rotation of $\boldsymbol{S}$ about this axis. The wall is assumed to move in response to an applied magnetic field Ω_0, and its motion is allowed to be damped through the inclusion of a *phenomenological* (Landau) damping factor α (Landau 1965).

The energy of magnetisation is written as $E = \int_{-\infty}^{+\infty} \mathrm{d}x\, \mathcal{H}(x)$, with

$$\mathcal{H}(x) = 2\pi S^2 \sin^2\theta \cos^2\phi - \tfrac{1}{2}J\tau S^2 \cos^2\theta + \Omega_0 S \cos\theta + \tfrac{1}{2}JS^2(\sin^2\theta\,\phi_x^2 + \theta_x^2). \tag{3.1}$$

The ensuing equations of motion are (introducing the phenomenological damping)

$$\theta_t - \alpha\phi_t \sin\theta = (\sin\theta)^{-1}[4\pi S \sin^2\theta \sin\phi\cos\phi - JS(\partial/\partial x)(\sin^2\theta\,\phi_x)], \tag{3.2}$$

$$\phi_t \sin\theta + \alpha\theta_t = 4\pi S \sin\theta\cos\theta\cos^2\phi + J\tau S \sin\theta\cos\theta - \Omega_0 \sin\theta + JS \sin\theta\cos\theta\,\phi_x^2 - JS\theta_{xx}. \tag{3.3}$$

If we assume that $\phi_x = 0 = \phi_t$, then $\phi = \phi_0 = \text{constant}$. Then from (3.2) we have

$$\theta_t = 4\pi S \sin\theta \sin\phi_0 \cos\phi_0, \tag{3.4}$$

$$\theta_{tt} = 4\pi S \cos\theta \sin\phi_0 \cos\phi_0 \theta_t = (4\pi S \cos\phi_0 \sin\phi_0)^2 \sin\theta\cos\theta. \tag{3.5}$$

Substituting (3.4) into (3.3) gives

$$\alpha 4\pi S \sin\theta \sin\phi_0 \cos\phi_0 = (4\pi S\cos^2\phi_0 + J\tau S)\sin\theta\cos\theta - \Omega_0 \sin\theta - JS\theta_{xx}. \tag{3.6}$$

Then (3.5) in (3.6) gives

$$\sin\theta(4\pi S\alpha \sin\phi_0\cos\phi_0 + \Omega_0) = JS(c_0^{-2}\theta_{tt} - \theta_{xx}), \tag{3.7}$$

where

$$c_0^2 = JS(4\pi S\cos\phi_0 \sin\phi_0)^2/[4\pi S\cos^2\phi_0 + J\tau S]. \tag{3.8}$$

We now look for solitary-wave solutions $\theta = \theta(x - \nu t)$. The only way to satisfy wall boundary conditions is to set the left hand side of (3.7) equal to zero. Then

$$\Omega_0/\alpha = -4\pi S \sin\phi_0 \cos\phi_0 \tag{3.9}$$

and

$$\theta_{tt} = (\Omega_0/\alpha)^2 \sin\theta\cos\theta. \tag{3.10}$$

It now follows from (3.6) that

$$\theta_{xx} - (JS)^{-1}(\alpha/\Omega_0)^2(4\pi S\cos^2\phi_0 + J\tau S)\theta_{tt} = 0. \tag{3.11}$$

However, (3.9) gives

$$2\cos^2\phi_0 = 1 \pm [1 - (\Omega_0/2\alpha\pi S)^2]^{1/2}, \tag{3.12}$$

so that we can re-express (3.8) as

$$\frac{1}{c_0^2} = \frac{\alpha^2\tau}{\Omega_0^2}\left[\!\left[\frac{2\pi}{J\tau}\left\{1 - \left[1 - \left(\frac{\Omega_0}{2\pi\alpha S}\right)^2\right]^{1/2}\right\} + 1\right]\!\right], \tag{3.13}$$

where we have chosen the negative square root to agree with the solution given in § 2.1 when $\Omega_0 \to 0$ ($\Omega_0/\alpha c_0 \to \tau^{1/2}$). The solution becomes imaginary for $\Omega_0 > 2\pi\alpha S$.

From (3.11)

$$c_0^2\theta_{xx} - \theta_{tt} = 0. \tag{3.14}$$

This is the wave equation and has solutions $\theta = \theta(x \pm c_0 t)$. However, we have to satisfy (3.14) and (3.4) simultaneously where a solution $\theta = \theta(x - \nu t)$ is sought to (3.4). To be consistent we require $\nu = c_0$, whence the dynamic domain wall solution is (cf § 2.1)

$$c_0\theta_x = (\Omega_0/\alpha)\sin\theta, \tag{3.15}$$

i.e.

$$\cos\theta = \tanh[(\Omega_0/\alpha c_0)(x - c_0 t)]. \tag{3.16}$$

(*b*) *Enz's Solution* (1964)

Enz introduced several drastic assumptions which, however, have the attractive feature of leading to the much studied dynamic sine–Gordon equation (Scott *et al* 1973, Bullough *et al* 1978). His assumptions were: (i) $\alpha = 0$ (no damping); (ii) $|\phi| \ll 1$; (iii) terms such as ϕ_x^2 in (3.3) can be neglected. Then, from (3.2) and retaining only terms to $O(\phi)$, we have

$$\theta_t = 4\pi S \sin\theta \sin\phi \cos\phi \simeq 4\pi S \sin\theta\phi, \tag{3.17}$$

$$\theta_{tt} = 4\pi S \sin\theta\phi_t. \tag{3.18}$$

Inserting (3.18) in (2.8) with $\alpha = 0$, but including a pole term of the form $-2\pi S^2 \sin^2\theta \cos^2\phi$, gives

$$\theta_{tt} = -(4\pi S)^2 \sin\theta\cos\theta - 4\pi J\tau S^2 \sin\theta\cos\theta - 4\pi S\Omega_0 \sin\theta + 4\pi JS^2\theta_{xx}. \tag{3.19}$$

Equation (3.19) is similar to that appearing in § 2.2. (Interestingly (3.19) is formally identical to the 'double sine–Gordon' equation appearing in descriptions of resonant optical media (Bullough *et al* 1978), excitations in the B-phase of superfluid ^{3}He (Maki and Kumar 1976), etc.) As in § 2.2 there are no π-domain walls unless $\Omega_0 = 0$, when

$$\theta_{xx} - (1/4\pi JS^2)\theta_{tt} = (\sin 2\theta/8\pi JS^2)[(4\pi S)^2 + 4\pi J\tau S^2]. \tag{3.20}$$

Equation (3.20) reduces to the static result (2.18) in the limit $4\pi JS^2 \gg 1$, as it should.

Result (3.20) is the very familiar sine–Gordon equation (Scott *et al* 1973) which interestingly also belongs to the class of systems totally integrable by means of the inverse scattering transform. Thus arbitrary solutions can again be constructed. The single-domain wall corresponds to a single soliton and is readily computed. We set $c_1^2 = 4\pi JS^2$ and seek a solitary-wave solution with wall boundary conditions. The 'pseudo-relativistic' ($c_1 \leftrightarrow$ 'speed of light') generalisation of (2.17) follows easily:

$$\cos\theta(x, t) = \tanh[\zeta(x - vt)], \tag{3.21}$$

with

$$\zeta^2 = \frac{1}{4\pi JS^2} \frac{(4\pi S)^2 + 4\pi J\tau}{1 - v^2/c_1^2}. \tag{3.22}$$

From (3.17) and (3.21) we find the requirement $\sin 2\phi = -2\zeta/4\pi S$. Thus this procedure is self-consistent provided that

$$|\zeta| \ll 2\pi S. \tag{3.23}$$

The conclusion from these two approaches is that a single, non-accelerating domain wall solution is only possible in the presence of a magnetic field if $\phi_t = \phi_x = 0$. In Enz's solution (3.21), ϕ is in fact allowed to vary (but is required to be small); however, the solution is immediately invalidated by a magnetic field. The conditions of validity are rather restrictive, and, since only free wall motions can be accommodated, the interesting experimental observations (see de Leeuw 1977) of field-dependent wall velocities cannot be addressed. It is physically clear that to achieve a terminal wall velocity we need damping in addition to the field (cf Fogel *et al* 1977, Bishop 1978a), and Walker's procedure shows how the appropriate velocity derives from a balance of these two affects. It will be interesting to reproduce Walker's solution (at least for small fields) by treating the field and damping terms as *perturbations* to the exact sine–Gordon soliton (3.21). This is also true for the effects of defects and other parameter inhomogeneities (Bishop 1978a, McLoughlin and Scott 1978).

Note that in Walker's solution the wall velocity must be less than a 'critical' value given by (3.13), depending on the relative magnitudes of damping and applied field. The solution is, however, forced from a single solitary-wave *ansatz* for a strongly driven nonlinear equation. The nominal critical velocity is larger than observed experimentally (de Leeuw 1977), and it is amusing to speculate that more complex nonlinear excitations may in fact appear, especially near the critical velocity. Such behaviour has been postulated near thresholds in other driven nonlinear systems (Nakajima *et al* 1974). In view of our knowledge of the pulse soliton and 2π-wall solutions of § 2.4, we might, for example, expect wall break-up, annihilation, or slowing down through the creation of finite-anisotropy pulses (cf soliton 'tailing' in the driven damped sine–Gordon system (Nakajima *et al* 1974) or vortex tailing in turbulent flows). Recall that the large-amplitude pulses have low velocity. These intriguing possibilities await further numerical investigation.

4. Conclusions and discussion

In this work we have examined excitations in the continuous-spin, classical Heisenberg ferromagnetic chain, with and without (uniaxial) anisotropy. We have limited our present analysis to *single, solitary-wave* solutions, i.e. with a single translational velocity aspect (except for certain intrinsic uniform angular rotations—see (2.23)). Thus we have not considered N solitary waves or more general solutions. We note, however, that the pulse soliton, obtained with zero anisotropy (§ 2.3), required $\phi \neq$ constant. This solution could be continuously deformed for finite anisotropy (§ 2.4), but in that case there is also the possibility of a *different* sector of domain wall excitations; but these were only possible with a constant velocity in an external field if damping was introduced and if $\phi =$ constant (§ 3(a)). This incompatibility suggests that if these modes can exist simultaneously (as a time-dependent composite excitation), then a nontrivial (nonlinear) interaction will be operative. It will be especially interesting to study this possibility numerically, particularly regarding collision processes (including damping and an applied field).

Complete solubility appears to be restricted to the isotropic limit, where the inverse spectral transform (IST) formalism is applicable, yielding a complete solution to the arbitrary initial value problem. Several authors (Lakshmanan 1977, Lamb 1976, 1977) have recently noted how the complete solubility follows by mapping the evolutions of energy and momentum densities onto the nonlinear Schrödinger equation, and subsequently deconvoluting the energy–momentum spectrum to yield excitations in the spin variables themselves. (This approach is directly related (Lamb 1977) to the motion of a special helical curve and is equivalent, for example, to the analysis of vortex filament motions in an incompressible inviscid fluid (Hasimoto 1972). A variety of related techniques follow (Lakshmanan 1978, Zhakarov and Manakov 1979), e.g. pseudopotentials, prolongation structures, fibre bundle theory, reduction to action–angle variable form, formulation as a Riemann problem, etc. The most direct relationship with the IST framework has been given recently by Takhtajan (1977), who showed how the nonlinear equations of motion for the spin variables *themselves* can be associated with an auxiliary linear scattering operator of the (two-component) Zhakarov–Shabat type, with an evolution operator of suitable (Lax) form in the inverse scattering procedure (Bullough *et al* 1978). In this way the *general* solution may be expressed in N-soliton form with all the remarkable properties associated with such completely integrable Hamiltonian systems (Bullough *et al* 1978)—e.g. the existence of an infinite number of constants of the motion, of which field energy, momentum and magnetisation are three in the present case (see § 2). It should be emphasised that all such properties are strictly valid only for the infinite, one-dimensional, continuum model with boundary conditions $\phi_x \to 0 (\bmod 2\pi)$ as $|x| \to \infty$. Interestingly the nonlinear Schrödinger equation also governs modulations of the *linear* spin waves if only *weak* nonlinearity is included as a perturbation (Corones 1977).

It is a simple matter (Takhtajan 1977) to incorporate an applied field in the isotropic Heisenberg ferromagnet while retaining the IST technique. However, it has not yet proven possible to apply this technique in the presence of anisotropy fields necessary for domain walls. It *is* possible to formulate a perturbation theory with respect to the isotropic limit, by considering the motion of eigenvalues of the linear scattering operator introduced in the IST theory (see Newell 1978). Such an approach is being studied presently. Clearly it is topologically impossible that the *single* pulse soliton and single domain wall could be continuously deformed into each other: they are quite distinct solution types with different boundary conditions. However, it is possible that pulse solitons with amplitude $\sim \pi (V \sim 0)$ will deform continuously into a wall–anti-wall pair in the presence of anisotropy and appropriate perturbations. Further analysis of these speculations and of general nonlinear solutions awaits the results of detailed numerical and analytical (e.g. perturbation) studies. Again we have not investigated the stability of the excitations, although linear stability analysis for the single solitary waves considered here is operationally straightforward (Scott *et al* 1973), since we have their analytic form. (Stability is guaranteed for $\tau = 0$.)

Our ultimate purpose in studying the elementary excitations in the continuum spin Heisenberg system is to use them to formulate a configurational phenomenology at finite temperatures (i.e. statistical mechanics), as has been done with other nonlinear systems (see Bishop 1978b). This would serve to emphasise *physically* distinct excitation contributions, particularly to static and dynamic response functions, and would be of practical and conceptual importance both experimentally and theoretically. These features are unlikely to be adequately represented in conventional linear theories and finite-order perturbations in linear modes. Similarly, conventional interpretations of experimental response probes of magnetic structure (neutron, x-ray, etc) are based on linear (or weakly anharmonic) prejudices and will have to be modified to be diagnostically receptive to possible local or other nonlinear excitations. Many theories of dynamic response functions in (one-dimensional) nonlinear models (Krumhansl and Schrieffer 1975, Kawasaki 1976, Mikeska 1978) have only partial validity, since they omit the crucial features of mode interactions in these non-superpositional problems. In future work we will formulate a configurational approach to the (static and dynamic) statistical mechanics of the isotropic continuum Heisenberg ferromagnetic chain, using the separability into nonlinear normal modes (available via IST) as has been done elsewhere (Bishop 1978b) for other separable nonlinear systems.

Accurate information is especially relevant in this area in view of the experimental accessibility of (and continuing activity in) quasi-one-dimensional spin systems (Steiner *et al* 1976, Kjems and Steiner 1978). Classical ferromagnetic systems are available†.

† Here we are emphasising the nonlinear interest in a ferromagnetic isotropic *Heisenberg* chain in a magnetic field. There is already complementary interest in the nonlinearity exhibited by one-dimensional *planar* ferromagnets (e.g. $CsNiF_3$) in a suitable field (Kjems and Steiner 1978). This system is related to a sine–Gordon-like model and exhibits strong soliton features in its statistical mechanics, since there is a gap in the excitation spectrum. The gapless spectrum of nonlinear modes in the present isotropic case ((2.24)) will be more subtle—cf breather excitations in the sine–Gordon system (Stoll *et al* 1979), although there gaps do remain, depending on the breather frequency. The anisotropy of § 2.4 also induces a gap, but for $K > 0$ there is Ising not XY symmetry.

Even more numerous are examples corresponding closely to one-dimensional classical Heisenberg *antiferromagnets* (Steiner *et al* 1976). Unfortunately the IST procedure does not seem to extend directly to this case. We note, however, that a recent study (Büttner and Bilz 1978) of the acoustic and optic *phonon* branches in a one-dimensional model with two atoms per unit cell has shown that each branch can be approximately associated with a different nonlinear equation (the modified Korteweg–de Vries and ϕ-four equations respectively (see Bullough *et al* 1978)). In this case it was necessary to retain terms to quartic order in a continuum representation. A similar approach may also be profitable for the antiferromagnetic spin system.

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Notes added in proof. Since the completion of this work we have received a preprint (Fogedby 1978) which also exploits the complete integrability of the classical isotropic continuum Heisenberg chain. Following the work of Takhtajan (1977), Fogedby exhibits the separable Hamiltonian structure (in continuum and soliton modes) in terms of generalised action–angle variables, as advocated in §§ 2.3 and 4. As we have indicated, this provides the starting point of a natural configurational representation for the statistical mechanics of the system, an approach which we shall describe in detail in a subsequent article.

Also, very recently it has been shown explicitly (Zhakarov and Manakov 1979) that the present isotropic Heisenberg system is a simple natural reduction of a generalised *matrix* nonlinear Schrödinger equation, which is itself completely integrable (cf comments in the text).

References

Bishop A R 1978a in *Solitons in Action* ed. K Lonngren and A C Scott (New York: Academic)
—— 1978b in *Solitons and Condensed Matter Physics, Springer Series in Solid State Physics*, vol. 8, ed. A R Bishop and T Schneider (Berlin: Springer)
Bishop A R, Domany E and Krumshansl J A 1977 *Ferroelectrics* **16** 183–6
Bullough R K *et al* 1978 various contributions to *Solitons and Condensed Matter Physics, Springer Series in Solid State Physics*, vol. 8, ed. A R Bishop and T Schneider (Berlin: Springer)
Büttner H and Bilz H 1978 in *Solitons and Condensed Matter Physics, Springer Series in Solid State Physics*, vol. 8, ed. A R Bishop and T Schneider (Berlin: Springer)
Corones J 1977 *Phys. Rev.* B **16** 1763–4
Enz U 1964 *Helv. Phys. Acta* **37** 245–51
Fogedby H C 1978 *Preprint*
Fogel M B *et al* 1977 *Phys. Rev.* B **15** 1578–92
Gradshteyn I S and Rhyzik I M 1965 in *Tables of Integrals, Series and Products* (New York: Academic) formula 2.266.
Hasimoto H 1972 *J. Fluid Mech.* **51** 477–85
Kaup D J and Newell A C 1978 *Proc. R. Soc.* **361** 413–46
Kawasaki K 1976 *Prog. Theor. Phys.* **55** 2029–30
Kjems J K and Steiner M 1978 *Phys. Rev. Lett.* **41** 1137–40
Krumhansl J A and Schrieffer J R 1975 *Phys. Rev.* B **11** 3535–45
Lakshmanan M 1977 *Phys. Lett. A* **61** 53–4
—— 1978 *Preprints*
Lakshmanan M, Ruijgrok Th W and Thompson C J 1976 *Physica* A **84** 577–90

Lamb G L Jr 1976 *Phys. Rev. Lett.* **37** 235–7 (erratum) 723
—— 1977 *J. Math. Phys.* **18** 1654–61
Landau L D 1965 in *Collected Papers of L D Landau* ed. D ter Haar (New York: Pergamon, Gordon and Breach), p101ff
de Leeuw F W 1977 *Physica* B **86–8** 1320–6
Luther A 1976 *Phys. Rev.* B **14** 2153–9
McLaughlin D W and Scott A C 1978 *Phys. Rev.* A **18** 1652–80
Maki K and Kumar P 1976 *Phys. Rev.* B **14** 3920–8
Mikeska H J 1978 *J. Phys. C: Solid St. Phys.* **11** L29–32
Nakajima K *et al* 1974 *J. Appl. Phys.* **45** 4095–9
Nakamura K and Sadada T 1974 *Phys. Lett.* A **48** 321–2
Newell A C 1978 in *Solitons and Condensed Matter Physics, Physics*, Vol. 8, ed. A R Bishop and T Schneider (Berlin: Springer)
Rubinstein J 1970 *J. Math. Phys.* **11** 258–66
Scott A C, Chu F Y F and McLaughlin D W 1973 *Proc. IEEE* **61** 1443–83
Steiner M, Villain J and Windsor C G 1976 *Adv. Phys.* **25** 87–209
Stoll E, Schneider T and Bishop A R 1979 *Phys. Rev. Lett.* **42** 937–40
Takhtajan L A 1977 *Phys. Lett.* A **64** 235–6
Tjon J and Wright 1977 *Phys. Rev.* B **15** 3470–6
Walker L R 1963 quoted by J F Dillon in *Magnetism* vol. III, ed. G T Rado and H Suhl (New York: Academic) p450
Zhakarov V E and Manakov S V 1979 *Preprint*
—— 1979 *Teor. Mat. Fiz.* at press